X a cpt R.S. $g \geq 2$

1. ~~Objects and Questions~~ / Big Picture

$\mathcal{M}_{Hod}$ = moduli of (polystable) λ-connections on rk n v.b $E \to X$ $(\Lambda^n E = X \times \mathbb{C})$

$$\mathcal{M}_{Hod} \ni [\lambda, \bar\partial_E, \nabla]$$
$$\downarrow \qquad\qquad \updownarrow$$
$$\mathbb{C} \qquad \lambda$$

$$\Omega^0(E) \xrightarrow{\bar\partial_E} \Omega^{0,1}(E) \qquad \bar\partial_E(fs) = \bar\partial f \otimes s + f\bar\partial_E s.$$
$$\oplus$$
$$\Omega^0(E) \xrightarrow{\nabla} \Omega^{1,0}(E) \qquad \nabla(fs) = \lambda\partial f \otimes s + f\nabla s$$

$$(\bar\partial_E + \nabla)^2 = 0.$$

$\mathbb{C}^*$-action

$$\xi \cdot [\lambda, \bar\partial_E, \nabla] = [\xi\cdot\lambda, \bar\partial_E, \xi\nabla] \qquad \pi^{-1}(\lambda) \cong \pi^{-1}(\lambda\cdot\xi)$$

$\rightsquigarrow$ $\pi^{-1}(0)$ is special fiber.

Thm (Simpson) $\lim_{\xi\to 0} \xi\cdot[\lambda, \bar\partial_E, \nabla]$ always exists (and is a fixed point in $\pi^{-1}(0)$)

iso classes, so (Fixed $\not\Rightarrow \nabla = 0$)

B.B. Strat. $\rightsquigarrow$ $\mathcal{M}_{Hod} = \bigsqcup_{\alpha \in \pi_0(Fix)} W_\alpha$ $\pi^{-1}(\lambda) \cong \mathcal{M}^\lambda_{Hod} = \bigsqcup W^\lambda_\alpha$.

$\lambda = 0$ Higgs bundles

$$\pi^{-1}(0) = \mathcal{M}_H \xrightarrow{\ \mathcal{T}_1\ } \mathcal{M}_{dR} = \pi^{-1}(1)$$

hol-connections
hol sym. (J, Ω_J).

$\nabla = \Phi \in H^0(End_0(E) \otimes K)$

hol sym (I, Ω_I)

$$[0, \bar\partial_E, \Phi] \longmapsto [1, \bar\partial_E + \Phi^{*h}, \partial^h_E + \Phi].$$

$\exists$ h a metric.

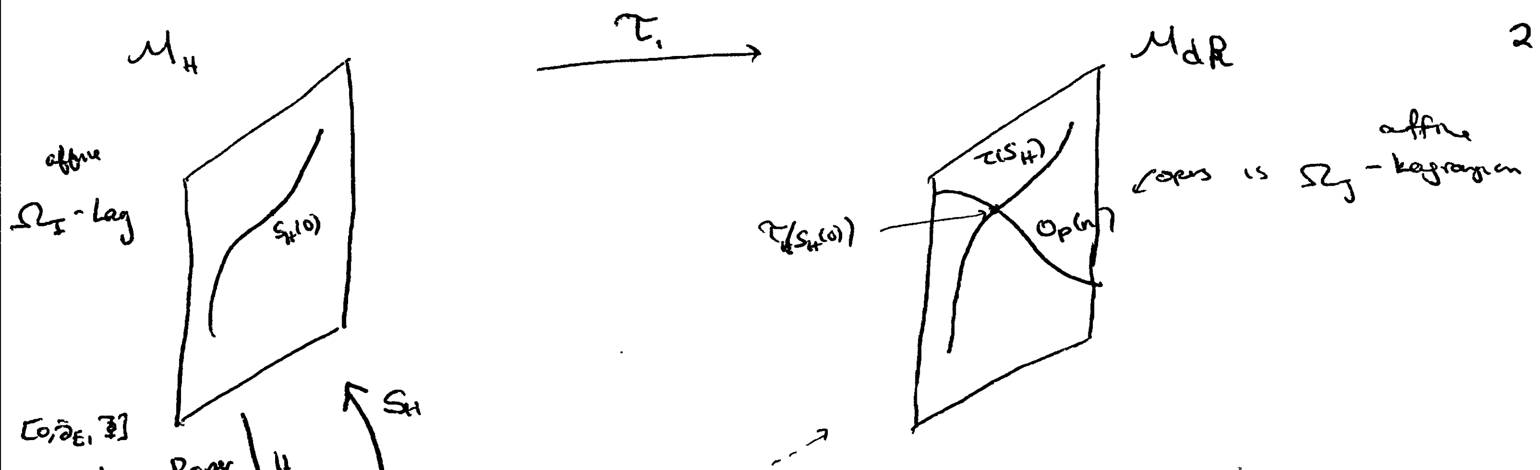
$\mu \in \Omega^1(End(E))$. $\Omega_J(\mu,\nu) = \int_X \mathrm{Tr}(\mu\wedge\nu)$.

$\boxed{\text{In case. } \Omega_I((\beta,\psi),(\alpha,\varphi)) = i\int_X \mathrm{Tr}(\varphi\wedge\beta - \psi\wedge\alpha)}$

$IJ = K$ $\rightsquigarrow$ hyperkähler

$$\mathbb{C} \times \mathcal{M}_H \xrightarrow{\ \mathcal{T}_c \text{ home}\ } \mathcal{M}_{Hod}$$
$$(\lambda, [0, \bar\partial_E, \Phi]) \longmapsto [\lambda, \bar\partial_E + \lambda\Phi^{*h}, \lambda\partial^h_E + \Phi].$$

$\mathcal{M}_{Hod}$ foliated by <u>sections</u>



$$[0,\bar\partial_E,\Phi] \downarrow \qquad \bigoplus_{j=2}^{n} H^0(K^j)$$

Charpoly(Φ). $\quad$ 1/2 dim

$\boxed{Ex}$ $\quad n=2 \qquad S_H(q_2) = [0, K^{1/2}\oplus K^{-1/2}, \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix}]$

when $q_2 = 0$. $\quad \tau(S_H(0)) = [1, \begin{pmatrix} \bar\partial_{K^{1/2}} & 1^* \\ & \bar\partial_{K^{-1/2}} \end{pmatrix}, \begin{pmatrix} \partial^h_{K^{1/2}} & 0 \\ 1 & \partial^h_{K^{-1/2}} \end{pmatrix}]$

$\tau_{\Phi}(S_H)$ = hol of hyperbolic structures

$$Op(q) = \{[1, \begin{pmatrix} \bar\partial_{K^{1/2}} & 1^* \\ 0 & \bar\partial_{K^{-1/2}} \end{pmatrix}, \begin{pmatrix} \partial^h_{K^{1/2}} & q_2 \\ 1 & \partial^h_{K^{-1/2}} \end{pmatrix}]$$

$\rightsquigarrow$ holonomies of $\mathbb{CP}^1$ structure on X.

3rd map

$e\mathcal{L}$. Thm (C W) $e\mathcal{L}$ identifies ~~hol~~ hol Lagrangians of the stratification ~~strata~~

2. $\mathcal{M}_H$ - stratification

<u>Fixed points</u>

$[0, \bar\partial_E, \Phi] = [0, \bar\partial_E, \zeta\Phi]$ for all $\zeta \in \mathbb{C}^*$ $\iff$

$$\bar\partial_E = \begin{pmatrix} \bar\partial_1 & & 0 \\ & \ddots & \\ 0 & & \bar\partial_\ell \end{pmatrix} \qquad \mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_\ell$$

$$\Phi = \begin{pmatrix} 0 & & & \\ \varphi_1 & \ddots & & 0 \\ & \ddots & & \\ & & \varphi_{\ell-1} & 0 \end{pmatrix} \qquad \varphi_i : \mathcal{E}_i \to \mathcal{E}_{i+1} \otimes K.$$

$$g_\zeta = \begin{pmatrix} Id_1 & & & \\ & Id_2\zeta & & \\ & & \ddots & \\ & & & Id_\ell \zeta^{\ell-1} \end{pmatrix} \qquad g_\zeta^{-1} \Phi g_\zeta = \zeta^{-1}\Phi.$$

$\mathcal{M}_H = \bigsqcup_\alpha W^0_\alpha \qquad \pi^0_\alpha : W^0_\alpha \to Fix_\alpha \quad$ has affine hol. Lagrangian's as fibers. (v.b.)

open stratum. dense. $\quad W^0_{open} = T^*Bun(Sl_n) \downarrow Bun(Sl_n) \qquad [0, \bar\partial_E, \Phi]$ so that $\bar\partial_E$ is polystable

Not closed since if Φ is nilpotent, $\lim_{\zeta\to\infty} [0, \bar\partial_E, \zeta\Phi]$ exists. and is a fixed point.

Closed strata $\Leftrightarrow$ no upward nilpotent defs.

$$S_H = W^{\oplus}_{S(0)} \downarrow * \qquad S_H(0) = K^{n/2}\oplus K^{\frac{n-3}{2}}\oplus\cdots\oplus K^{\frac{1-n}{2}} \qquad \Phi = \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ & 1 & 0 \end{pmatrix}$$

3. Opers & ∂W^1_α strat.

$$\lim_{\xi\to 0} \xi[1,\bar\partial_E,\nabla] = S_H(0). \qquad Op(n) = W^1\text{'s}$$

Given a $[1,\bar\partial_E,\nabla]$ a hol. filtration $0=F_0\subset F_1\subset\cdots\subset F_\ell = E$ is Griffiths $\pitchfork$.

if $\nabla: F_i \to F_{i+1}\otimes K$.

$\rightsquigarrow$ Higgs bundle $\bar\partial_E \cong \begin{pmatrix} \bar\partial_{F_1/F_0} & & * \\ & \ddots & \\ & & \bar\partial_{F_\ell/F_{\ell-1}} \end{pmatrix}$ $\quad \nabla = \begin{pmatrix} \ddots & & * \\ \varphi_1 & \ddots & \\ & \varphi_{\ell-1} & \ddots \end{pmatrix}$.

$$(\bar\partial_0,\Phi_0) = \left(\begin{pmatrix} \bar\partial_1 & & \\ & \ddots & \\ & & \bar\partial_\ell \end{pmatrix}, \begin{pmatrix} & & 0 \\ \varphi_1 & & \\ & \varphi_{\ell-1} & \end{pmatrix}\right)$$

if polystable, then this is the limit. since $\left(g_\xi^{-1}\bar\partial_E g_\xi,\ g_\xi^{-1}\xi\nabla g_\xi\right)$

$$= \left(\begin{pmatrix} \bar\partial_1 & & \xi\to 0\ * \\ & \ddots & \\ & & \bar\partial_\ell \end{pmatrix}, \begin{pmatrix} & & *\xi\to 0 \\ \varphi_1 & & \\ & \varphi_{\ell-1} & \end{pmatrix}\right) \qquad \xi\to 0 \quad \checkmark$$

So opers have full filtration $F_1\subset\cdots\subset F_n$ &

$$\nabla: F_i/F_{i-1} \longrightarrow F_{i+1}/F_i\otimes K \quad \text{is an iso.}$$

Simpson $\Rightarrow$ can always find Griffiths $\pitchfork$ filtration so that the Higgs is p-s.

$$\mathcal{I}(W^0_\alpha) \not\subset W^1_\alpha, \quad \text{but} \quad \mathcal{T}_1(Fix_\alpha)\subset W^1_\alpha.$$

$$\left[0, \begin{pmatrix} \bar\partial_1 & & \\ & \ddots & \\ & & \bar\partial_\ell \end{pmatrix}, \begin{pmatrix} \varphi_1 & & \\ & \ddots & \\ & & \varphi_{\ell-1} \end{pmatrix}\right] \to \left[1, \begin{pmatrix} \bar\partial_1 & \varphi_1^* & \\ & \ddots & \varphi_{\ell-1}^* \\ & & \bar\partial_\ell \end{pmatrix}, \begin{pmatrix} \partial_1^h & & 0 \\ \varphi_1 & \ddots & \\ 0 & \varphi_{\ell-1} & \partial_\ell^h \end{pmatrix}\right]$$

4. Conformal limits

1-more gadget, conformal limit which biholomorphically identifies the fibers of W^{∞}_α w/ fibers of W^1_α.

$$\mathcal{CL}: \mathbb{C}^*\times\mathcal{M}_H \longrightarrow \mathcal{M}_{Hod}$$
$$(\hbar,[0,\bar\partial_E,\Phi]) \longrightarrow [\hbar, \mathcal{CL}^{\hbar}(\bar\partial_E,\Phi)].$$

$\mathcal{CL}^h$

$$[0, \bar\partial_E, \Phi] \xrightarrow{\ \tau_\lambda\ } [\lambda, \bar\partial_E + \lambda\Phi^{*_h}, \lambda\partial_E^h + \Phi]$$

$R>0$ $$[0, \bar\partial_E, R\Phi] \xrightarrow{\ \tau_\lambda\ } [\lambda, \bar\partial_E + \lambda R\Phi^{*_{h_R}}, \lambda\partial_E^{h_R} + R\Phi]$$

act by R^{-1}

$\hbar = R^{-1}\lambda$ $$[\hbar, \bar\partial_E + \hbar R^2\Phi^{*_{h_R}}, \hbar\partial_E^{h_R} + \Phi]$$

$$\mathcal{CL}^\hbar([0, \bar\partial_E, \Phi]) = \lim_{R\to 0}[\hbar, \bar\partial_E + \hbar R^2\Phi^{*_{h_R}}, \hbar\partial_E^{h_R} + \Phi].$$

Gaiotto conj. $\mathcal{CL}^\hbar(S_H) = Op(n)$ bihol. proven by DFKMMN

Theorem (C-Wentworth)

Let $(\bar\partial_0, \Phi_0) \in Fix_\alpha$ be stable, then for every $[0, \bar\partial_E, \Phi] \in W_\alpha^0(\bar\partial_0, \Phi_0)$,

$\mathcal{CL}^\hbar(0, \bar\partial_E, \Phi)$ is welldefined and.

$\mathcal{CL}^\hbar$ gives a bihol. ident. of $W_\alpha^0(\bar\partial_0, \Phi_0) \cong W_\alpha^\hbar(\bar\partial_0, \Phi_0)$.

Proof First Prove a global slice Theorem for fibes of $W_\alpha^\hbar$.

$S_+ \to H^1_L$ bihol

(Kuranishi map $S \to H^1$ local homeo

For $(\bar\partial_0, \Phi_0) \in Fix$ stable

$$S_+ \subset S = \left\{(\beta, \psi) \in \Omega^{0,1}(\mathrm{End}\,E) \oplus \Omega^{1,0}(E) \;\middle|\; \begin{array}{l}\bar\partial_0\psi + [\Phi_0, \beta] + [\psi, \beta] = 0 \\ \partial_0^{h_0}\beta + [\Phi_0^*, \psi] = 0\end{array}\right\}$$

$$S_+ \times \mathbb{C} \xrightarrow{\ \Psi\ } \mathcal{M}_{Hol}$$
$$(\beta, \psi, \lambda) \longmapsto [\lambda, \bar\partial_0 + \lambda\Phi_0^{*_{h_0}} + \beta, \lambda\partial_E^{h_0} + \Phi_0 + \psi]$$

$\Psi(S_+ \times \{\lambda\}) = W_\alpha^\lambda(\bar\partial_0, \Phi_0)$

$$\mathcal{CL}^\hbar(\Psi(\beta, \psi, 0)) = \Psi(\beta, \psi, \hbar).$$

$$W_\alpha^\lambda = \Psi(S_+ \times \{\lambda\}).$$

$$\boxed{\begin{array}{l} S_+ = \left\{(\beta, \psi) \in S \;\middle|\; \begin{array}{l}\beta \in N \\ \psi \in L \oplus N\end{array}\right\} \\ \bigoplus \\ \cancel{\Delta = \bigoplus \ldots,\ \mathrm{Hom}(E_j, E_{j+k})} \\ N_- \oplus L \oplus N_+ = \mathrm{End}(E) \end{array}}$$